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Distributional Borel summability of odd anharmonic oscillators

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Abstract. It is proved that the divergent Rayleigh–Schrödinger perturbation expansions for the eigenvalues of any odd anharmonic oscillator are Borel summable in the distributional sense to the resonances naturally associated with the system.

1. Introduction and statement of the results

Recent work on complex operators with a real spectrum (see, e.g., [3–5, 12–14, 22] and references therein) in quantum mechanics and on the so-called Bessis–Zinn Justin conjecture have generated renewed interest in the spectral and perturbation theory of odd anharmonic oscillators in quantum mechanics, namely the class of Schrödinger operators in $L^2(\mathbb{R})$ defined (on a domain to be specified later) by the action of the differential operator

$$H(\beta) = p^2 + x^2 + \beta x^{2k+1} \equiv H(0) + \beta x^{2k+1} \qquad k = 1, 2, \ldots . \tag{1.1}$$

Here $p = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$, β, the coupling constant, is a numerical parameter and k is fixed. The spectral and perturbation theory of the operators $H(\beta)$ (the first perturbation theory examples that were introduced in quantum mechanics, see, e.g., [6]) was settled long ago, from a mathematically rigorous standpoint, for non-real values of the coupling constant ([7]; see also [1, 2, 15]). The main results (see the summary for more details) are: if $\beta \in \mathbb{C}$, $\mathrm{Im}\,\beta > 0$ (analogous results for $\mathrm{Im}\,\beta < 0$) $H(\beta)$ has a discrete spectrum, and for any $j = 0, 1, \ldots,$ there is exactly one eigenvalue $E_j(\beta)$ near the eigenvalue $2j + 1$ of $H(0)$ for $|\beta|$ suitably small. The Rayleigh–Schrödinger perturbation expansion near $2j + 1$ exists to all orders, is even and is Borel (more precisely, Borel–Leroy of order $q \equiv (2k - 1)/2)$) summable to $E_j(\beta)$ for $\pi/8 + \eta < \arg\beta < 7\pi/8 - \eta$, $\eta > 0$ (see [4] for tests of numerical accuracy). Hence if β is purely imaginary and small the eigenvalues $E_j(\beta)$ are real.

These results, however, leave completely open the problem of the meaning of the perturbation series for $\beta \in \mathbb{R}$. In this case $H(\beta)$ does not define a unique self-adjoint operator; nevertheless the function $E_j(\beta)$ can be analytically continued to $\beta \in \mathbb{R}$ where it can still be interpreted as a resonance of the problem, as recalled in the summary. Its real part (which admits the perturbation series as an asymptotic expansion) represents the location of the resonance and its imaginary part represents the width. However, when $\beta \in \mathbb{R}$ the coefficients of this power series have constant sign; as is well known, this prevents Borel summability because the Borel transform develops a pole on the positive real axis.

The notion of distributional Borel summability (more precisely, in this case, Borel–Leroy of order q) was introduced in [8] exactly to deal with this kind of situations and its validity was proved in [10, 9] for the eigenvalues of the double-well oscillator and for the Stark effect resonances, respectively. It is recalled in the following:

Definition 1.1. *Let q be a rational number, $(a_s)_{s\in\mathbb{N}}$ a sequence of real numbers and $R > 0$. We say that the formal series $\sum_{s=0}^{\infty} a_s\beta^s$ is Borel–Leroy summable of order q in the distributional sense to $f(\beta)$ for $0 < \beta < R$ if the following conditions are satisfied.*

(a) Set

$$B(t) \equiv \sum_{s=0}^{\infty} \frac{a_s}{\Gamma(qs+1)} t^s. \tag{1.2}$$

Then $B(t)$ is holomorphic in some circle $|t| < \Lambda$; moreover $B(t)$ admits a holomorphic continuation to the intersection of some neighbourhood of $\mathbb{R}_+ \equiv \{t \in \mathbb{R} : t > 0\}$ with $\mathbb{C}^+ \equiv \{t \in \mathbb{C} : \operatorname{Im} t > 0\}$.

(b) The boundary value distribution $B(t + \mathrm{i}0)$ exists $\forall t \in \mathbb{R}_+$ and the following representation holds:

$$f(\beta) = \frac{1}{q\beta}\int_0^{\infty} PP(B(t))\,\mathrm{e}^{-(t/\beta)^{1/q}}\left(\frac{t}{\beta}\right)^{-1+1/q}\mathrm{d}t \tag{1.3}$$

for β belonging to the Nevanlinna disc of the $\beta^{1/q}$-plane $C_R \equiv \{\beta : \operatorname{Re}\beta^{-1/q} > R^{-1}\}$, where $PP(B(t)) = \frac{1}{2}(B(t+\mathrm{i}0) + \overline{B(t+\mathrm{i}0)})$.

If $q = 1$ the series is called Borel summable in the distributional sense to $f(\beta)$.

Remark 1.2.

(a) As for the ordinary Borel sum, the representation (1.3) is unique among all real functions admitting the prescribed formal power-series expansion and fulfilling suitable analyticity requirements and remainder estimates (the Nevanlinna conditions: see the appendix for their definition in the distributional case).

(b) In the definition of the Nevanlinna disc the principal determination is taken whenever an ambiguity is generated by the power $\beta^{-1/q}$.

In the case of the Stark effect the distributional Borel summability puts into one-to-one correspondence the perturbation series near the hydrogen bound states with the real part (location) of the resonances. Here the analogous result is proved for the odd anharmonic oscillators, namely,

Theorem 1.3. *Let $q = (2k-1)/2$, $j \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $f_j(\beta) \equiv \operatorname{Re} E_j(\beta)$, $g_j(\beta) \equiv \operatorname{Im} E_j(\beta)$. Then*

(a) The Rayleigh–Schrödinger perturbation expansion near $2j+1$ is Borel–Leroy summable of order q in the distributional sense to $f_j(\beta)$ for $|\beta|$ suitably small; in particular,

$$f_j(\beta) = \frac{1}{q|\beta|}\int_0^{\infty} PP(B_j(t))\,\mathrm{e}^{-(t/|\beta|)^{1/q}}\left(\frac{t}{|\beta|}\right)^{-1+1/q}\mathrm{d}t \tag{1.4}$$

where $B_j(t)$ is defined as in (1.2).

(b) $f_j(\beta) = f_j(-\beta)$, $g_j(\beta) = -g_j(-\beta)$.

Remark 1.4.

(a) The symmetry property $f_j(\beta) = f_j(-\beta)$ is a consequence of the property $a_{2l+1} = 0, \forall l$, which in turn follows from the odd symmetry of the perturbation x^{2k+1}.

(b) The distributional Borel summability in the direction $\arg\beta = \pi$ is reduced to definition 1.1 in theorem 3.13 below. In particular (remark 3.12), $E_j(\beta)$ turns out to be analytic also in the disc $C_{R'} = \{\beta : \mathrm{Re}\,(\beta')^{-1/q} > (R')^{-1}, \beta' = \beta \mathrm{e}^{-\mathrm{i}\pi}\}$, so that the following representation holds:

$$d_j(\beta) \equiv 2\mathrm{i} g_j(\beta) = \begin{cases} E_j(\beta) - \overline{E_j(\overline{\beta})} & \beta \in C_R \\ \overline{E_j(\overline{\beta})} - E_j(\beta) & \beta \in C_{R'}. \end{cases} \tag{1.5}$$

Here $d_j(\beta)$ is the 'discontinuity', which has zero asymptotic expansion (see the appendix).

(c) The distributional Borel summability procedure actually also determines the imaginary part of the functions $E_j(\beta)$, $\beta \in \mathbb{R}$, i.e. also the width of the resonances. The discussion of this aspect is postponed after the proof of theorem 1.3.

(d) The analyticity domain specified in theorem 2.1 below allows a direct application of the Harrell–Simon argument [17] relating the imaginary part of the resonance, divergence of the perturbation expansion and WKB barrier penetration formula. One has, for the ground state (i.e. $j = 0$)

$$\mathrm{Im}\, E_j(\beta) \sim \mathrm{e}^{-A/\beta^{1/q}} \qquad A \equiv \int_0^1 u\sqrt{1-u^{2q}}\,\mathrm{d}u. \tag{1.6}$$

This is a straightforward computation, completely analogous to those of [16, 17] for the quartic anharmonic oscillators, the double-well and the Stark effect. The details are omitted.

The proof of theorem 1.3 requires the verification of the analogue of the Nevanlinna criterion, stated and proved in theorem 4 of [8], and recalled here in the appendix for the convenience of the reader. The proof of the criterion is accomplished in two steps. In the first one (details in section 2) it is proved that the eigenvalues $E_j(\beta)$, $\mathrm{Im}\,\beta > 0$, admit a (many-valued) analytic continuation to a (Riemann surface) sector wider than that obtained in [7], namely $-(2k-1)\pi/4 < \arg\beta < (2k+3)\pi/4$. To do this we apply to this situation the Hunziker–Vock technique [18], developed after [7], to establish eigenvalue stability. The second one (section 3) consists in extending this analyticity to a suitable Nevanlinna disc, as required by the criterion for distributional Borel summability. We do this by adapting to the present situation the techniques introduced in [10, 9] to deal with the double-well oscillators and the Stark effect. Finally, the interpretation of the present and previous mathematical results in terms of quantum mechanical resonances is described in the summary.

2. Analytic continuation of the complex eigenvalues

Let $k \in \mathbb{N}$ be fixed and $\beta \in \mathbb{C} - \{0\}$; $H(\beta)$ will denote the operator in $L^2(\mathbb{R})$ defined by $D(H(\beta)) = D(p^2) \cap D(x^{2k+1})$ and

$$H(\beta)u = (p^2 + x^2 + \beta x^{2k+1})u \qquad \forall u \in D(H(\beta)). \tag{2.1}$$

In [7] it was proved that, for $\mathrm{Im}\,\beta > 0$, $H(\beta)$ represents a holomorphic family of type A of operators with compact resolvents and, for $|\beta|$ suitably small, a non-empty (discrete) spectrum. The norm resolvent convergence of $H(\beta)$ to the harmonic oscillator

$$H(0) = p^2 + x^2 \qquad D(H(0)) = D(p^2) \cap D(x^2) \tag{2.2}$$

as $|\beta| \to 0$, $\mathrm{Im}\,\beta > 0$, yielded the stability of the eigenvalues of $H(0)$ with respect to the family $H(\beta)$ in the following sense: for any fixed $j \in \mathbb{N}$ and $\forall\delta > 0$, there exists $B_j(\delta) \equiv B(\delta) > 0$

such that for $|\beta| < B(\delta)$, $\operatorname{Im}\beta > 0$, $H(\beta)$ has exactly one eigenvalue $E_j(\beta)$ such that $|E_j(\beta) - (2j+1)| < \delta$, and therefore $E_j(\beta) \to (2j+1)$ as $|\beta| \to 0$, $\operatorname{Im}\beta > 0$. Moreover, such eigenvalues are analytic functions of β, for $|\beta| < B(\delta)$, $\operatorname{Im}\beta > 0$, and they admit a (many-valued) analytic continuation across the real axis to the (Riemann surface) sector

$$S_1(\delta) = \left\{\beta : |\beta| < B(\delta), -(2k-1)\tfrac{1}{8}\pi + \delta < \arg\beta < (2k+7)\tfrac{1}{8}\pi - \delta\right\}. \tag{2.3}$$

Finally, there exist constants $C, \eta > 0$ such that the corresponding Rayleigh–Schrödinger perturbation expansion is Borel summable to $E_j(\beta)$ in the sector $|\beta| < C, \pi/8 + \eta < \arg\beta < 7\pi/8 - \eta$. The main result in this section consists in extending the analyticity of the eigenvalues of $H(\beta)$ to the wider sector $-(2k-1)\pi/4 + \delta < \arg\beta < (2k+3)\pi/4 - \delta$, as stated in the following

Theorem 2.1. *The eigenvalues $E_j(\beta)$ of $H(\beta)$, $\operatorname{Im}\beta > 0$, which exist for $|\beta|$ suitably small, admit a (many-valued) analytic continuation across the real axis to any sector*

$$S(\delta) = \left\{\beta : |\beta| < B(\delta), -(2k-1)\tfrac{1}{4}\pi + \delta < \arg\beta < (2k+3)\tfrac{1}{4}\pi - \delta\right\}, \forall\delta > 0. \tag{2.4}$$

In order to prove this theorem we need some preliminary results based on the standard method of dilation analyticity (see, e.g., [20], vol IV, section XIII.10). More precisely we introduce the operator

$$H(\beta,\theta) \equiv \mathrm{e}^{-2\theta}p^2 + \mathrm{e}^{2\theta}x^2 + \beta\mathrm{e}^{(2k+1)\theta}x^{(2k+1)} \equiv \mathrm{e}^{-2\theta}K(\beta,\theta) \tag{2.5}$$

which, for $\theta \in \mathbb{R}$, is unitarily equivalent to $H(\beta)$, $\operatorname{Im}\beta > 0$, via the dilation operator $U(\theta)$ defined by

$$(U(\theta)u)(x) = \mathrm{e}^{\theta/2}u(\mathrm{e}^{\theta}x) \qquad \forall u \in L^2(\mathbb{R}).$$

In [7] it was proved that, when defined on $D(p^2) \cap D(x^{2k+1})$, $H(\beta,\theta)$ represents a holomorphic family of type A of operators with compact resolvents for $-(2k-1)\pi/8 < \arg\beta < (2k+7)\pi/8$, $\operatorname{Im}\theta = (\pi/2 - \arg\beta)/(2k+3)$. This was obtained by means of a quadratic estimate for the operator $p^2 + \mathrm{e}^{4\theta}x^2 + \mathrm{i}|\beta|x^{2k+1}$ (which corresponds to $K(\beta,\theta)$ for $\arg\beta + (2k+3)\operatorname{Im}\theta = \pi/2$), valid for $-\pi/2 < 4\operatorname{Im}\theta < \pi/2$. Now, a first step in the proof of theorem 2.1 consists in proving an analogous quadratic estimate for the operator

$$K(\beta,\theta) = p^2 + \mathrm{e}^{4\theta}x^2 + |\beta|\mathrm{e}^{\mathrm{i}\arg\beta + (2k+3)\theta}x^{2k+1} \tag{2.6}$$

under two more general conditions

$$\begin{aligned} &0 < \arg\beta + (2k+3)\operatorname{Im}\theta < \pi \\ &0 < \arg\beta + (2k-1)\operatorname{Im}\theta < \pi. \end{aligned} \tag{2.7}$$

Remark 2.2. The first condition of (2.7) corresponds to requiring the positivity of the imaginary part of the coefficient of x^{2k+1}; as for the second one, if we denote $\alpha = \arg\beta + (2k+3)\operatorname{Im}\theta$ the argument of the coefficient of x^{2k+1}, it is equivalent to requiring that the coefficient $\gamma \equiv \mathrm{e}^{4\theta}$ of x^2 is in the half-plane $-\pi + \alpha < \arg\gamma < \alpha$.

Lemma 2.3. *Let $\alpha \in\,]0,\pi[$ and $\Omega \subset \mathbb{C}$ be a compact subset of the half-plane $-\pi + \alpha < \arg\gamma < \alpha$. Then there exist $a, b > 0$ such that*

$$\|p^2u\|^2 + |\gamma|^2\|x^2u\|^2 + |\beta|^2\|x^{2k+1}u\|^2 \leqslant a\|(p^2 + \gamma x^2 + |\beta|\mathrm{e}^{\mathrm{i}\alpha}x^{2k+1})u\|^2 + b\|u\|^2 \tag{2.8}$$

$\forall u \in D(p^2) \cap D(x^{2k+1})$, $\gamma \in \Omega$, $0 < |\beta| \leqslant 1$, *a and b independent of γ in Ω and α in a closed interval contained in* $]0,\pi[$.

Proof. We shall prove the following estimate, equivalent to (2.8):

$$\|p^2u\|^2 + |\sigma|^2\|x^2u\|^2 + |\beta|^2\|x^{2k+1}u\|^2 \leqslant a\|(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + \sigma x^2 + |\beta|x^{2k+1})u\|^2 + b\|u\|^2 \qquad (2.9)$$

$\forall u \in D(p^2) \cap D(x^{2k+1})$, with $\sigma = \gamma \mathrm{e}^{-\mathrm{i}\alpha}$ varying in a compact subset of the half-plane $-\pi < \arg\sigma < 0$. As quadratic forms on $D(p^2) \cap D(x^{2k+1}) \otimes D(p^2) \cap D(x^{2k+1})$ we have

$$\begin{aligned}
&(\mathrm{e}^{\mathrm{i}\alpha}p^2 + \overline{\sigma}x^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + \sigma x^2 + |\beta|x^{2k+1})\\
&\quad= (\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) + |\sigma|^2x^4 + \mathrm{Re}\,\sigma(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})x^2\\
&\qquad+\mathrm{i}\,\mathrm{Im}\,\sigma(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})x^2 + \mathrm{Re}\,\sigma x^2(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})\\
&\qquad-\mathrm{i}\,\mathrm{Im}\,\sigma x^2(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})\\
&\quad= \left|\frac{\mathrm{Re}\,\sigma}{\sigma}\right|(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1} \pm |\sigma|x^2)(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1} \pm |\sigma|x^2)\\
&\qquad+\left(1 - \left|\frac{\mathrm{Re}\,\sigma}{\sigma}\right|\right)\left[(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) + |\sigma|^2x^4\right]\\
&\qquad+\mathrm{i}\,\mathrm{Im}\,\sigma(\mathrm{e}^{\mathrm{i}\alpha}p^2x^2 - \mathrm{e}^{-\mathrm{i}\alpha}x^2p^2)\\
&\quad\geqslant \left(1 - \left|\frac{\mathrm{Re}\,\sigma}{\sigma}\right|\right)\left[(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) + |\sigma|^2x^4\right]\\
&\qquad+\mathrm{i}\,\mathrm{Im}\,\sigma\cos\alpha[p^2, x^2] - \mathrm{Im}\,\sigma\sin\alpha(p^2x^2 + x^2p^2)\\
&\quad= \left(1 - \left|\frac{\mathrm{Re}\,\sigma}{\sigma}\right|\right)\left[(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) + |\sigma|^2x^4\right]\\
&\qquad+2\,\mathrm{Im}\,\sigma\cos\alpha(px + xp) - \mathrm{Im}\,\sigma\sin\alpha\left([p,[p,x^2]] + 2px^2p\right)\\
&\quad= \left(1 - \left|\frac{\mathrm{Re}\,\sigma}{\sigma}\right|\right)\left[(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) + |\sigma|^2x^4\right]\\
&\qquad-2\,\mathrm{Im}\,\sigma|\cos\alpha|(\mp px \mp xp) - \mathrm{Im}\,\sigma\sin\alpha(-2 + 2px^2p)
\end{aligned}$$

(since $\sin\alpha > 0$ and $\mathrm{Im}\,\sigma < 0$)

$$\begin{aligned}
&\quad\geqslant \left(1 - \left|\frac{\mathrm{Re}\,\sigma}{\sigma}\right|\right)\left[(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) + |\sigma|^2x^4\right]\\
&\qquad-2\,\mathrm{Im}\,\sigma|\cos\alpha|\left[(p \mp x)^2 - p^2 - x^2\right] + 2\,\mathrm{Im}\,\sigma\sin\alpha\\
&\quad\geqslant \left(1 - \left|\frac{\mathrm{Re}\,\sigma}{\sigma}\right|\right)\left[(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) + |\sigma|^2x^4\right]\\
&\qquad+2\,\mathrm{Im}\,\sigma|\cos\alpha|(p^2 + x^2) + 2\,\mathrm{Im}\,\sigma\sin\alpha.
\end{aligned}$$

In [7] it was proved that there exist $a_1, b_1 > 0$, in general depending on $|\beta|$, such that

$$(\mathrm{e}^{\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + |\beta|x^{2k+1}) \geqslant a_1(p^4 + |\beta|^2x^{4k+2}) - b_1.$$

Thus,

$$\begin{aligned}
&(\mathrm{e}^{\mathrm{i}\alpha}p^2 + \overline{\sigma}x^2 + |\beta|x^{2k+1})(\mathrm{e}^{-\mathrm{i}\alpha}p^2 + \sigma x^2 + |\beta|x^{2k+1})\\
&\quad\geqslant A(p^4 + |\beta|^2x^{4k+2}) + B|\sigma|^2x^4 + 2\,\mathrm{Im}\,\sigma|\cos\alpha|(p^2 + x^2) + 2\,\mathrm{Im}\,\sigma\sin\alpha - b_1\\
&\quad\geqslant [Aa'p^4 + 2\,\mathrm{Im}\,\sigma|\cos\alpha|p^2 + 2\,\mathrm{Im}\,\sigma\sin\alpha - b + b'/2]\\
&\qquad+[Aa'|\beta|^2x^{4k+2} + 2\,\mathrm{Im}\,\sigma|\cos\alpha|x^2 + b'/2]\\
&\qquad+A(1 - a')p^4 + A(1 - a')|\beta|^2x^{4k+2} + B|\sigma|^2x^4 - b'.
\end{aligned}$$

Now it suffices to choose $0 < a' < 1$ and $b' > 0$ such that the two terms in square brackets are positive. $\square$

Lemma 2.4. *Let β and θ be fixed, satisfying conditions (2.7) and let $\alpha = \arg\beta + (2k+3)\,\mathrm{Im}\,\theta$, $\alpha \in\,]0, \pi[$. Then there exists $\xi > 0$ such that*

$$\xi\,\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}(\alpha-\pi/2)}\,\langle u, K(\beta,\theta)u\rangle\right] \geqslant \langle u, p^2 u\rangle \qquad \forall u \in C_0^\infty(\mathbb{R}). \tag{2.10}$$

Proof. We have

$$\begin{aligned}
&\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}(\alpha-\pi/2)}\left\langle u, (p^2 + \mathrm{e}^{4\theta}x^2 + |\beta|\mathrm{e}^{(2k+3)\,\mathrm{Re}\,\theta+\mathrm{i}\alpha}x^{2k+1})u\right\rangle\right] \\
&\qquad = \cos(\alpha-\pi/2)\langle u, p^2u\rangle + \mathrm{e}^{4\,\mathrm{Re}\,\theta}\cos(\pi/2 - \alpha + 4\,\mathrm{Im}\,\theta)\langle u, x^2u\rangle \\
&\qquad\quad + |\beta|\mathrm{e}^{(2k+3)\,\mathrm{Re}\,\theta}\cos(\pi/2)\langle u, x^{2k+1}u\rangle \\
&\qquad = \sin\alpha\langle u, p^2u\rangle + \mathrm{e}^{4\,\mathrm{Re}\,\theta}\sin(\alpha - 4\,\mathrm{Im}\,\theta)\langle u, x^2u\rangle \\
&\qquad \geqslant \sin\alpha\langle u, p^2u\rangle
\end{aligned}$$

since $\sin(\arg\beta + (2k-1)\,\mathrm{Im}\,\theta) > 0$ by the second condition of (2.7). Moreover, since $0 < \alpha < \pi$, the lemma is proved with $\xi = (\sin\alpha)^{-1}$. $\square$

Theorem 2.5. *Let $s = \arg\beta$ and $t = \mathrm{Im}\,\theta$. Then $H(\beta,\theta)$ is a holomorphic family of type A of closed operators on $D(H(\beta,\theta)) = D(p^2)\cap D(x^{2k+1})$ with compact resolvents for β and θ such that s and t vary in the parallelogram P of the (s,t)-plane defined by*

$$P = \{(s,t) \in \mathbb{R}^2 : 0 < (2k-1)t + s < \pi, 0 < (2k+3)t + s < \pi\}. \tag{2.11}$$

Proof. Lemma 2.3 guarantees that $H(\beta,\theta)$ is closed on a domain independent of β and θ for $\arg\beta = s$ and $\mathrm{Im}\,\theta = t$ satisfying conditions (2.7):

$$0 < (2k+3)t + s < \pi$$

$$0 < (2k-1)t + s < \pi$$

which define the parallelogram P with vertices in the points of coordinates $(-(2k-1)\pi/4, \pi/4)$, $(0,0)$, $((2k+3)\pi/4, -\pi/4)$, $(\pi, 0)$. From lemma 2.4 it follows that, for β and θ in this region, $K(\beta,\theta)$ has a numerical range in the half-plane $-\pi + \alpha \leqslant \arg z \leqslant \alpha$, with $\alpha = \arg\beta + (2k+3)\,\mathrm{Im}\,\theta$; thus $H(\beta,\theta)$ has a numerical range contained in the half-plane

$$\Pi = \{z \in \mathbb{C} : -\pi + \arg\beta + (2k+1)\,\mathrm{Im}\,\theta \leqslant \arg z \leqslant \arg\beta + (2k+1)\,\mathrm{Im}\,\theta\}.$$

By standard arguments on the holomorphic families of type A (see [19] or [20], vol IV), taking into account the above-mentioned results obtained in [7] for $-(2k-1)\pi/8 < \arg\beta < (2k+7)\pi/8$, we now obtain the analyticity of $H(\beta,\theta)$ in the region defined by P, which allows β to be extended to the sector $-(2k-1)\pi/4 < \arg\beta < (2k+3)\pi/4$, as well as the compactness of the resolvents. Finally, the (discrete) spectrum of $H(\beta,\theta)$ is contained in Π and $\forall z \notin \Pi$, $\|(z - H(\beta,\theta))^{-1}\| \leqslant (\mathrm{dist}(z,\Pi))^{-1}$. $\square$

Remark 2.6. Let us note that, if we start from the operator $H(\beta)$ with $\mathrm{Im}\,\beta < 0$, analogous results can be obtained for the operator family $H(\beta,\theta)$ for β and θ such that $s = \arg\beta$, $t = \mathrm{Im}\,\theta$ vary in the parallelogram

$$P^1 = \{(s,t) \in \mathbb{R}^2 : -\pi < (2k-1)t + s < 0, -\pi < (2k+3)t + s < 0\}.$$

Furthermore, the adjoint operator $H(\beta,\theta))^*$ of $H(\beta,\theta)$ is $H(\overline{\beta},\overline{\theta})$.

In order to complete the proof of theorem 2.1 we need to extend to the wider sector $S(\delta)$ given by (2.4) the result obtained in [7] for $\beta \in S_1(\delta)$ (see (2.3)), on the existence of eigenvalues of $H(\beta, \theta)$ and on their convergence to the corresponding eigenvalues of the harmonic oscillator as $|\beta| \to 0$. To this end, since we cannot make use of the norm resolvent convergence which holds only for $\beta \in S_1(\delta)$, $|\beta| \to 0$, we will apply the more general criterion for the stability of the eigenvalues introduced in [18] and based on the strong convergence of the resolvents. More precisely, let us consider the operator

$$H(0,\theta) \equiv \mathrm{e}^{-2\theta} p^2 + \mathrm{e}^{2\theta} x^2 \qquad D(H(0,\theta)) = D(p^2) \cap D(x^2)$$

corresponding to the dilated harmonic oscillator. We will prove that the eigenvalues of $H(0, \theta)$, independent of θ for $-\pi/4 < \mathrm{Im}\,\theta < \pi/4$, and represented by the sequence of the odd numbers $\{(2j + 1) : j \in \mathbb{N}\}$, are stable in the sense of Kato with respect to the family $\{H(\beta, \theta) : |\beta| > 0\}$, β and θ in the region defined by P. For simplicity we will work with the operators $K(\beta, \theta) = \mathrm{e}^{2\theta} H(\beta, \theta)$ and $K(0, \theta) = \mathrm{e}^{2\theta} H(0, \theta)$; moreover, from now on we will assume θ to be purely imaginary, that is of the form $\mathrm{i}\theta$, $-\pi/4 < \theta < \pi/4$, and (with a slight abuse of notation) we will still denote by $H(\beta, \theta)$ and $K(\beta, \theta)$ the operators $H(\beta, \mathrm{i}\theta)$ and $K(\beta, \mathrm{i}\theta)$, respectively. Note that with this convention we should read θ in place of $\mathrm{Im}\,\theta$ wherever the notation $\mathrm{Im}\,\theta$ has been employed, in particular in the conditions (2.7). Finally, let $\sigma(K(\beta, \theta))$ denote the spectrum of $K(\beta, \theta)$. Then, in order to obtain the above-mentioned stability result, we will prove the following:

Theorem 2.7. *Let β and θ satisfy conditions (2.7). We have*

(a) if $\lambda \notin \sigma(K(0, \theta))$, then $\lambda \in \Delta$, where

$$\Delta = \{z \in \mathbb{C} : z \notin \sigma(K(\beta,\theta)) \text{ and } (z - K(\beta,\theta))^{-1} \textit{ is uniformly bounded as } |\beta| \to 0\};$$

(b) if $\lambda \in \sigma(K(0, \theta)) = \{(2j + 1)\,\mathrm{e}^{2\mathrm{i}\theta} : j \in \mathbb{N}\}$, then λ is stable with respect to the family $K(\beta, \theta)$, i.e. if $r > 0$ is sufficiently small, so that the only eigenvalue of $K(0, \theta)$ enclosed in $\Gamma_r = \{z \in \mathbb{C} : |z - \lambda| = r\}$ is λ, then there is $B > 0$ such that for $|\beta| < B$, $\dim P(\beta, \theta) = \dim P(0, \theta)$, where

$$P(\beta,\theta) = (2\pi \mathrm{i})^{-1} \oint_{\Gamma_r} (z - K(\beta,\theta))^{-1}\,\mathrm{d}z$$

is the spectral projection of $K(\beta, \theta)$ corresponding to the part of the spectrum enclosed in $\Gamma_r \subset \mathbb{C} - \sigma(K(\beta, \theta))$. Similarly for $P(0, \theta)$.

Proof. It is a straightforward application of theorem 5.4 of [18] once we have proved the following:

Theorem 2.8. *Let $\arg\beta$ and θ be fixed, satisfying conditions (2.7), and let $K(\rho) = K(\beta, \theta)$ with $\rho = |\beta|$. Then*

(a) $\lim_{\rho\to 0^+} K(\rho)u = K(0)u$, $\lim_{\rho\to 0^+} K(\rho)^*u = K(0)^*u$, $\forall u \in C_0^\infty(\mathbb{R})$.

(b) $\Delta \neq \emptyset$.

(c) Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\chi(x) = 1$ for $|x| \leqslant 1$, $0 \leqslant \chi(x) \leqslant 1$, $\forall x \in \mathbb{R}$, $\chi(x) = 0$ for $|x| \geqslant 2$. For $n \in \mathbb{N}$ let $\chi_n(x) = \chi(x/n)$ and $M_n(x) = 1 - \chi_n(x)$. We have

1. *if $\rho_m \to 0^+$ and $u_m \in D(K(\rho_m))$ are two sequences such that*

$$\|u_m\| \to 1 \quad u_m \xrightarrow{w} 0 \qquad \textit{and} \qquad \|K(\rho_m)u_m\| \leqslant (\textit{constant}) \quad \forall m$$

then there exists $a > 0$ such that

$$\limsup_{m\to\infty} \|M_n u_m\| \geqslant a > 0 \qquad \forall n$$

2. *for some* $z \in \Delta$

$$\lim_{n\to\infty} \|[M_n, K(\rho)](z - K(\rho))^{-1}\| = 0$$

uniformly as $\rho \to 0^+$;

3. $\forall \lambda \in \mathbb{C}$, *there exists* $\delta > 0$ *such that*

$$d_n(\lambda, \rho) \equiv \inf\{\|(\lambda - K(\rho))M_n u\| : u \in D(K(\rho)), \|M_n u\| = 1\} > \delta$$

$\forall n > n_0$ *and* $\rho \to 0^+$.

Proof.

(a) It follows immediately from the convergence of the potential $V(\rho) = e^{4i\theta}x^2 + \rho e^{i(\arg\beta+(2k+3)\theta)}x^{2k+1}$ to $V(0) = e^{4i\theta}x^2$ as $\rho \to 0^+$, uniformly on the compact subsets of $\mathbb{R}$.

(b) As already observed in the proof of theorem 2.5 $K(\rho)$ has a numerical range contained in the half-plane

$$\Pi_\alpha = \{z \in \mathbb{C} : -\pi + \alpha \leqslant \arg z \leqslant \alpha\} \qquad \alpha = \arg\beta + (2k+3)\theta$$

independent of ρ, and $\forall z \notin \Pi_\alpha$, $\|(z - K(\rho))^{-1}\| \leqslant (\mathrm{dist}(z, \Pi_\alpha))^{-1}$.

(c) Statement 1 follows from a standard argument based on an estimate which comes from lemma 2.4: there exists $c > 0$ such that

$$\|(1 + p^2)^{\frac{1}{2}}u\| \leqslant c(\|K(\rho)u\| + \|u\|) \qquad \forall u \in D(K(\rho)). \tag{2.12}$$

For the details see [18]. As for statement 2, following again [18], we have

$$[M_n, K(\rho)] = [\chi_n, p^2] = 2in^{-1}\Phi_n p - n^{-2}\Psi_n$$

where the functions Φ_n and Ψ_n, obtained by differentiating χ once and twice, respectively, are uniformly bounded in n and ρ. Thus, the result follows by applying (2.12) again. Finally, given $\lambda \in \mathbb{C}$ we have

$$d_n(\lambda, \rho) = \inf\{\|(\lambda' - e^{i(\pi/2-\alpha)}K(\rho))M_n u\| : u \in D(K(\rho)), \|M_n u\| = 1\}$$

with $\lambda' = e^{i(\pi/2-\alpha)}\lambda$, $\alpha = \arg\beta + (2k+3)\theta$. Therefore, $d_n(\lambda, \rho) \geqslant \mathrm{dist}(\lambda', G_n(\rho))$, where

$$G_n(\rho) = \left\{\left\langle M_n u, e^{i(\pi/2-\alpha)}K(\rho))M_n u\right\rangle : u \in D(K(\rho)), \|M_n u\| = 1\right\}$$

whence

$$d_n(\lambda, \rho) \geqslant \inf\left\{\mathrm{Re}\left\langle M_n u, e^{i(\pi/2-\alpha)}K(\rho))M_n u\right\rangle - |\lambda'| : u \in D(K(\rho)), \|M_n u\| = 1\right\}.$$

Now the assertion follows from the proof of lemma 2.4, which yields

$$\mathrm{Re}\left\langle M_n u, e^{i(\pi/2-\alpha)}K(\rho))M_n u\right\rangle$$
$$\geqslant \sin(\arg\beta + (2k-1)\theta)\langle M_n u, x^2 M_n u\rangle \geqslant n^2 \sin(\arg\beta + (2k-1)\theta)$$

and therefore

$$\lim_{\substack{n\to\infty \\ \rho\to 0^+}} d_n(\lambda, \rho) = +\infty. \qquad \square$$

Remark 2.9. It can be immediately checked that all the results so far obtained, in particular the analyticity of the family $H(\beta,\theta)$ and the stability of the eigenvalues of the harmonic oscillator with respect to $H(\beta,\theta)$ as $\rho = |\beta| \to 0^+$, hold uniformly in β and θ such that $(\arg\beta, \theta)$ varies in any compact subset of P.

Proof of theorem 2.1. It follows from theorems 2.5 and 2.7 and from remark 2.9. In particular, if $(\arg\beta, \theta) \in P$, by the well known Symanzik scaling properties (see [21]) the eigenvalues $E_j(\beta)$ of $H(\beta,\theta)$ do not depend on θ and represent the analytic continuation to the sector $S(\delta)$ of the eigenvalues of $H(\beta)$, $\operatorname{Im}\beta > 0$; in fact, as already observed, the condition $(\arg\beta, \theta) \in P$, allows us to extend $\arg\beta$ to the interval $]-(2k-1)\pi/4, (2k+3)\pi/4[$. □

Remark 2.10. Let $E_j(\beta)$ denote the generic eigenvalue of $H(\beta)$ for $\operatorname{Im}\beta > 0$, which can be analytically continued to the sector $S(\delta)$, and $E_j^1(\beta)$ the generic eigenvalue of $H(\beta)$ for $\operatorname{Im}\beta < 0$, which can be analytically continued to the sector

$$\overline{S}(\delta) = \left\{\beta : 0 < |\beta| < B(\delta), -(2k+3)\tfrac{1}{4}\pi + \delta < \arg\beta < (2k-1)\tfrac{1}{4}\pi - \delta\right\}.$$

Then, from remark 2.6 we have $E_j^1(\beta) = \overline{E_j(\overline{\beta})}$.

3. Analyticity of the eigenvalues in a Nevanlinna disc and distributional Borel summability

We begin this section by stating and proving the basic analyticity result needed to establish the distributional Borel summability (see the appendix).

Theorem 3.1. *Set $q = (2k-1)/2$. For each eigenvalue $E_j(\beta)$, $j \in \mathbb{N}$, of the odd anharmonic oscillator $H(\beta)$ there exists $R > 0$ such that $E_j(\beta)$ is analytic in the Nevanlinna disc $C_R = \{\beta : \operatorname{Re}\beta^{-1/q} > R^{-1}\}$ of the $\beta^{1/q}$-plane.*

Remark 3.2.

(I) The sector $S(\delta)$ can be rewritten in terms of the parameter q:

$$S(\delta) = \left\{\beta : |\beta| < B(\delta), -\frac{\pi}{2} + \frac{\delta}{q} < \arg\beta^{1/q} < \frac{\pi}{2} + \frac{\pi}{q} - \frac{\delta}{q}\right\}.$$

(II) The function $E_j(\beta)$, analytic in any sector $S(\delta)$ and for which we want to prove analyticity in a disc C_R, represents an eigenvalue of the operator $H(\beta,\theta)$ if the pair (β,θ) satisfies the condition $(\arg\beta, \theta) \in P$. In particular, for $-\pi(2k-1)/4 < \arg\beta < 0$ we can choose the path inside P given by the straight line of equation

$$\theta = -\frac{1}{2k+1}\arg\beta + \frac{\pi}{2(2k+1)}$$

then, if we set

$$\arg\beta = -\tfrac{1}{4}\pi(2k-1) + \tfrac{1}{2}\epsilon(2k-1) = -\tfrac{1}{2}\pi q + \epsilon q$$

i.e.

$$\arg\beta^{1/q} = -\tfrac{1}{2}\pi + \epsilon \qquad \epsilon \to 0^+$$

we obtain $\theta = \pi/4 - (2k-1)\epsilon/[2(2k+1)] = \pi/4 - \epsilon q/(2k+1)$, and the operator $H(\beta,\theta)$ takes the form

$$A(\rho) = \mathrm{e}^{-\mathrm{i}(\pi/2-(2k-1)\epsilon/(2k+1))}p^2 + \mathrm{e}^{\mathrm{i}(\pi/2-(2k-1)\epsilon/(2k+1))}x^2 + \mathrm{i}\rho x^{2k+1} \qquad \text{with} \quad \rho = |\beta|.$$

(III) For $\beta = \rho e^{i \arg \beta}$ and $\arg \beta = (-\frac{1}{2}\pi + \epsilon)q$, the boundary of C_R has the equation

$$\sin \epsilon = \frac{\rho^{1/q}}{R}. \tag{3.1}$$

Since the disc C_R can be regarded as the union of the boundaries of discs of smaller radius, the proof of theorem 3.1 reduces to a stability argument with respect to the family $A(\rho)$, as $\rho \to 0^+$, under condition (3.1), for the eigenvalues of a suitable limiting operator, which we proceed to define.

The argument is similar to that already developed in [9, 10] to obtain analyticity of the eigenvalues for the operators associated with the Stark effect and the double-well oscillators, respectively. More precisely, let D denote the dense subset of $L^2(\mathbb{R})$ of the functions which are translation analytic in a suitable strip $|\operatorname{Im} x| < \eta_0$, for some $0 < \eta_0 < 1$ (recall that $u \in L^2(\mathbb{R})$ is translation analytic for $|\operatorname{Im} x| < r$ if $(T_a u)(x) = u(x+a)$ admits an L^2-valued analytic continuation to $|\operatorname{Im} a| < r$); D represents a core for $A(\rho)$.

Definition 3.3. *Let $\eta > 0$ be fixed and small. For fixed $a_k > 0$, set $x_0 = -a_k/\rho^{1/(2k-1)}$ and let $\mathcal{U}$ denote the unitary operator in $L^2(\mathbb{R})$ defined by*

$$(\mathcal{U}\psi)(x) = (\xi_\rho'(x))^{1/2}\psi(\xi_\rho(x)) \qquad \forall \psi \in D$$

where, for any given $\rho > 0$, $\xi_\rho \in C^\infty(\mathbb{R})$ satisfies the conditions

$$\begin{aligned} &\xi_\rho(x) = x - i\eta \arctan\left[x/(1+x^2)^{1/4}\right] && -x_0 \leqslant x < +\infty \\ &\xi_\rho(x) = x && x \leqslant x_0 - \eta \end{aligned} \tag{3.2}$$

and $\operatorname{Im} \xi_\rho(x)$ *is monotonic in the remaining region.*

Then the closed operator $H_\rho \equiv \mathcal{U}A(\rho)\mathcal{U}^{-1}$, unitarily equivalent to $A(\rho)$ and with the same (discrete) spectrum, has $D_1 \equiv \mathcal{U}(D)$ as a core, and its action on D_1 is given by

$$\begin{aligned} H_\rho u = {} & \exp\left[-i\left(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon\right)\right]\left\{p f_\rho^2 p + 4^{-1}(f_\rho^2)''\right\} u \\ & + \exp\left[i\left(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon\right)\right]\xi_\rho^2 u + i\rho\xi_\rho^{2k+1} u \qquad \forall u \in D_1 \end{aligned} \tag{3.3}$$

where $f_\rho(x) = (\xi_\rho'(x))^{-1}$, $\forall x \in \mathbb{R}$.

Remark 3.4. In a similar way we can define the dilated harmonic oscillator, having D_1 as a core:

$$H_0 u = -i\left\{p f_0^2 p + 4^{-1}(f_0^2)''\right\} u + i\xi_0^2 u \qquad \forall u \in D_1$$

where $f_0(x) = (\xi_0'(x))^{-1}$ and $\xi_0'(x) = x - i\eta \arctan\left[x/(1+x^2)^{1/4}\right]$, $\forall \in \mathbb{R}$. In corollary 3.9 we will prove that H_0 is the limit in the strong resolvent sense of H_ρ as $\rho \to 0^+$. Therefore, as anticipated after remark 3.2, the proof of theorem 3.1 consists in obtaining a stability result for the eigenvalues $E_j = (2j+1)$, $j \in \mathbb{N}$, of H_0, which coincide with those of the harmonic oscillator, with respect to the family H_ρ as $\rho \to 0^+$.

Proceeding in analogy with [9, 10], this result will be obtained by proving some preliminary lemmas aimed at verifying the hypotheses of theorem A.1 of [10]. This theorem represents a simpler tool for applications, in the context of the more general stability theory developed by Hunziker and Vock in [18]. In particular, in the subsequent lemmas 3.5, 3.6, 3.9, 3.10 and corollaries 3.7 and 3.8, we follow the corresponding steps used in [9, 10] to obtain similar results, each one adapted to the specific characteristics of the present problem; we will describe here the relevant details.

Lemma 3.5. *Let* $V_\rho(x) = \exp\left[\mathrm{i}\left(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon\right)\right]\xi_\rho^2(x) + \mathrm{i}\rho\xi_\rho^{2k+1}(x)$. *Then for a suitable choice of the constant* $a_k > 0$ *in definition 3.3 there exist constants* $c_1 > 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\operatorname{Re} V_\rho(x) \geqslant \frac{c_1}{R} + c_2 \qquad \forall x \notin (-n, n) \tag{3.4}$$

$\forall n \geqslant n_0$, $0 < \rho < \rho_0$.

Proof. Set $\eta(x) = \operatorname{Im}\xi_\rho(x)$; then $\eta(x) \leqslant 0$ for $x > 0$, $\eta(x) \geqslant 0$ for $x \leqslant 0$, and $-\eta\pi/2 \leqslant \eta(x) \leqslant \eta\pi/2$, $\forall x \in \mathbb{R}$. Now a simple calculation gives

$$\begin{aligned}\operatorname{Re} V_\rho(x) = {} & \sin\{\epsilon(2k-1)/(2k+1)\}\left(x^2 - \eta(x)^2\right) - \cos\{\epsilon(2k-1)/(2k+1)\}\,(2x\eta(x))\\ & -\rho\eta(x)\left[(2k+1)x^{2k} - \binom{2k+1}{3}x^{2k-2}\eta(x)^2 + \binom{2k+1}{5}x^{2k-4}\eta(x)^4\right.\\ & \left. + \cdots + (-1)^{k-1}\binom{2k+1}{2k-1}x^2\eta(x)^{2k-2} + (-1)^k\eta(x)^{2k}\right].\end{aligned} \tag{3.5}$$

Next we note that the term inside the square brackets can be bounded from below by a constant (independent of ρ), and for $x \geqslant n \geqslant n_0$, $0 < \rho < \rho_0$ we have $x^2 > \eta(x)^2$, whence

$$\operatorname{Re} V_\rho(x) \geqslant cn + c' \geqslant \frac{c_1}{R} + c_2. \tag{3.6}$$

For $x \leqslant -n$ we still have $x^2 > \eta(x)^2$, and the term inside the square brackets in (3.5) can be bounded from above by

$$Ax^{2k} + B$$

for suitable constants $A > 0$ and $B \in \mathbb{R}$, independent of ρ and n. Thus,

$$\begin{aligned}\operatorname{Re} V_\rho(x) \geqslant {} & \sin[\epsilon(2k-1)/(2k+1)]\left(x^2 - \eta(x)^2\right) - \cos[\epsilon(2k-1)/(2k+1)]\,(2x\eta(x))\\ & -\rho\eta(x)(Ax^{2k} + B).\end{aligned} \tag{3.7}$$

Now, if the number $a_k > 0$ in definition 3.3 is chosen so that the polynomial term

$$-2x\cos[\epsilon(2k-1)/(2k+1)] - \rho(Ax^{2k} + B) \tag{3.8}$$

attains its (positive) maximum at $x_0 = -a_k/\rho^{1/(2k-1)}$, estimate (3.6) still holds in the interval $x_0 \leqslant x \leqslant -n$, if we make the assumption, not restrictive in this context, that $n \ll \rho^{-2k}$. Finally, note that at some point smaller than x_0 the term (3.8) becomes negative and tends to $-\infty$ as $\rho \to 0^+$, without being compensated by the term

$$\sin[\epsilon(2k-1)/(2k+1)]\left(x^2 - \eta(x)^2\right)$$

which behaves as $\frac{\rho^{1/q}}{R}x^2$, if we recall that $\sin\epsilon = \frac{\rho^{1/q}}{R}$. This is the reason why it was necessary to set $\eta(x) = 0$ for $x \leqslant x_0 - \eta$. In particular, in this region we have

$$\operatorname{Re} V_\rho(x) = (\sin[\epsilon(2k-1)/(2k+1)])\,x^2 \geqslant c\left(\frac{\rho^{1/q}}{R}\right)\left(-\frac{a_k}{\rho^{1/(2k-1)}} - \eta\right)^2 \geqslant \frac{c_1}{R} + c_2$$

whence the assertion. □

From now on the constant $a_k > 0$ in definition 3.3 will be chosen so as to satisfy lemma 3.5.

Lemma 3.6. *There exist constants* $c_3, c_4 > 0$ *such that*

$$\mathrm{Re}\,\langle u, H_\rho u\rangle \geqslant c_3 \int_{x_0}^{+\infty} \frac{(1+x^2)^{1/4}}{x^2+(1+x^2)^{1/2}} |pu|^2\,\mathrm{d}x - c_4\|u\|^2 \tag{3.9}$$

$\forall u \in D(H_\rho)$, $0 < \rho < \rho_0$.

Proof. Set $\omega = \exp\left[-\mathrm{i}\left(\frac{\pi}{2} - \frac{2k-1}{2k+1}\epsilon\right)\right]$. Then we have

$$\mathrm{Re}\,\langle u, H_\rho u\rangle = \mathrm{Re}\int_{-\infty}^{+\infty} \left\{\omega f_\rho^2 |pu|^2 + \tfrac{1}{4}\omega (f_\rho^2)''|u|^2 + V_\rho(x)|u|^2\right\} \mathrm{d}x. \tag{3.10}$$

As for the first term in the right-hand side of (3.10) we have

$$\mathrm{Re}\,(\omega f_\rho^2) = \sin[\epsilon(2k-1)/(2k+1)]\,\mathrm{Re}\, f_\rho^2 + \cos[\epsilon(2k-1)/(2k+1)]\,\mathrm{Im}\, f_\rho^2. \tag{3.11}$$

For $x \geqslant x_0$ it is easy to check that

$$\mathrm{Re}\, f_\rho^2 \geqslant \frac{1}{4}\left(1 - \eta^2 \frac{(1+x^2)^{1/2}}{[x^2+(1+x^2)^{1/2}]^2}\right) \tag{3.12}$$

and

$$\mathrm{Im}\, f_\rho^2 \geqslant \eta \left[\frac{(1+x^2)^{1/4}}{x^2+(1+x^2)^{1/2}}\right] \tag{3.13}$$

whence

$$\mathrm{Re}\,(\omega f_\rho^2) \geqslant \eta \left(\cos[\epsilon(2k-1)/(2k+1)] \frac{(1+x^2)^{1/4}}{x^2+(1+x^2)^{1/2}}\right). \tag{3.14}$$

In the region $x \leqslant x_0 - \eta$ we have $f_\rho(x) = 1$, so that

$$\mathrm{Re}\,(\omega f_\rho^2) = \sin[\epsilon(2k-1)/(2k+1)]. \tag{3.15}$$

Now simple calculations allow us to verify that $|(f_\rho^2)''|$ is bounded. Moreover, from (3.5) it follows that $\mathrm{Re}\, V_\rho(x)$ is bounded from below in the interval $(-n_0, n_0)$, and therefore in $\mathbb{R}$ by lemma 3.5. Now the assertion follows by combining this result with (3.14) and (3.15). □

Corollary 3.7.

(a) $\lim_{\rho\to 0^+} H_\rho u = H_0 u, \forall u \in D_1$.

(b) $\Delta' \neq \emptyset$, *where*

$$\Delta' = \{z \in \mathbb{C} : z \notin \sigma(H_\rho) \text{ and } (z - H_\rho)^{-1} \textit{ is uniformly bounded as } \rho \to 0^+\}.$$

(c) H_ρ *converges strongly to* H_0 *in the generalized sense.*

Proof. Statement (a) follows from the fact that $\xi_\rho(x) \to \xi_0(x)$ as $\rho \to 0^+$, uniformly on compacts. By lemma 3.6 we have that the numerical range of H_ρ is contained in a right half-plane Π, and since H_ρ has discrete spectrum, $\|(z - H_\rho)^{-1}\| \leqslant (\mathrm{dist}(z, \Pi))^{-1}, \forall z \notin \Pi$. Finally, (c) follows from (a) and (b), since D_1 is a core for H_ρ, $\rho \geqslant 0$ (see [19], theorem VIII.1.5). □

Corollary 3.8. *Let $\chi \in C_0^\infty(\mathbb{R})$ be the function defined in theorem 2.8(c), and again let $\chi_n(x) = \chi(x/n)$, $M_n(x) = 1 - \chi_n(x)$, $\forall n \in \mathbb{N}$. Then there exists $c_5 > 0$ such that*

$$\|[H_\rho, \chi_n]u\| \leqslant \frac{c_5}{n^{1/4}}(\|H_\rho u\| + \|u\|) \tag{3.16}$$

$\forall u \in D(H_\rho)$, $0 \leqslant \rho < \rho_0$.

Proof. Let $u \in D(H_\rho)$, $\|u\| = 1$, and γ_{2n} be the characteristic function of the interval $[-2n, 2n]$. We have

$$[H_\rho, \chi_n] = \omega[pf_\rho^2 p, \chi_n] = \omega\gamma_{2n}\{2in^{-1} f_\rho^2 \chi'(x/n)p + 2n^{-1} f_\rho f_\rho' \chi'(x/n) + n^{-2} f_\rho^2 \chi''(x/n)\}. \tag{3.17}$$

Now, since χ', χ'', f_ρ, f_ρ', f_ρ^2 are all bounded functions, we have the pointwise estimate

$$|[H_\rho, \chi_n]u(x)| \leqslant \frac{c}{n}(|u(x)| + |(pu)(x)|). \tag{3.18}$$

Thus, for $\|u\| = 1$,

$$\begin{aligned}
\|[H_\rho, \chi_n]u\| &\leqslant \frac{c'}{n}\left\{\left(\int_{-2n}^{2n} |pu|^2 \frac{(1+x^2)^{1/4}}{x^2 + (1+x^2)^{1/2}} \frac{x^2 + (1+x^2)^{1/2}}{(1+x^2)^{1/4}}\,\mathrm{d}x\right)^{1/2} + 1\right\} \\
&\leqslant \frac{c''}{n}\left\{n^{3/4}\left(\int_{x_0}^{+\infty} |pu|^2 \frac{(1+x^2)^{1/4}}{x^2 + (1+x^2)^{1/2}}\,\mathrm{d}x\right)^{1/2} + 1\right\} \\
&\leqslant \frac{c_5}{n^{1/4}}\left\{\mathrm{Re}\,\langle u, H_\rho u\rangle + 1\right\}
\end{aligned}$$

whence the assertion. Note that to obtain the second inequality we assumed again, without loss, $n \ll |x_0|$, while for the last inequality we have used lemma 3.6. □

Lemma 3.9. *Let the sequences $\rho_m \to 0^+$ and $u_m \in D(H_{\rho_m})$ be given such that $\|H_{\rho_m} u_m\|$ is bounded, $\|u_m\| = 1$, $u_m \xrightarrow{w} 0$. Then $\forall n$*

$$\lim_{m\to\infty} \|\chi_n u_m\| = 0.$$

Proof. Set $H_\rho' = \omega^{-1} H_\rho$ and let $\lambda \in \mathbb{C} - \sigma(H_0')$ be fixed. Then we have

$$\|\chi_n u_m\|^2 \leqslant c\left(\|\chi_n R_0'(H_0' - H_{\rho_m}')u_m\|^2 + \|\chi_n R_0'(H_{\rho_m}' - \lambda)u_m\|^2\right)$$

where $R_0' = (\lambda - H_0')^{-1}$. Now we can proceed as in the proof of lemma 5 of [9]. □

Lemma 3.10. *For any $\lambda \in \mathbb{C}$ there exist R, n_0, $\delta > 0$ such that*

$$d_{n,\rho}(\lambda) \equiv \inf\{\|(\lambda - H_\rho)M_n u\| : u \in D(H_\rho), \|M_n u\| = 1\} \geqslant \delta$$

$\forall n > n_0$, $\forall \rho \leqslant \rho_0$.

Proof. By lemma 3.5

$$\mathrm{Re}\,\langle M_n u, V_{\rho_m} M_n u\rangle \geqslant \frac{c_1}{R} + c_2 > \delta > 0$$

if $\|M_n u\| = 1$ and R is chosen sufficiently small. Finally, from the proof of lemma 3.6 the kinetic part of H_ρ is bounded from below and this proves the lemma. □

Proof of theorem 3.1. From corollary 3.8 and lemmas 3.9 and 3.10, the proof of a theorem analogous to theorem 2.8 immediately follows, with the operator $K(\rho)$ replaced by H_ρ, $\rho \geqslant 0$. Thus, we can apply theorem A1 of [10], in order to obtain the following stability result:

(a) if $\lambda \notin \sigma(H_0)$ then $(\lambda - H_\rho)^{-1}$ is uniformly bounded as $\rho \to 0^+$;
(b) if $\lambda \in \sigma(H_0)$ then λ is a stable eigenvalue with respect to the family $\{H_\rho\}_{\rho>0}$. □

With an argument analogous to that used to prove theorem 3.1 we now obtain the following.

Theorem 3.11. *Let $q = (2k-1)/2$. Then for each eigenvalue $E_j(\beta)$, $j \in \mathbb{N}$, of $H(\beta)$, $\operatorname{Im}\beta > 0$, there exists $R' > 0$ such that $E_j(\beta)$ is analytic in the Nevanlinna disc*

$$D_{R'} = \left\{\beta : |\beta^{1/q} - (R'/2)\,\mathrm{e}^{\mathrm{i}\pi/q}| < R'/2, -\frac{\pi}{2} + \frac{\pi}{q} < \arg\beta^{1/q} < \frac{\pi}{2} + \frac{\pi}{q}\right\}$$

with radius $R'/2$ and centre at $C = (R'/2)\,\mathrm{e}^{\mathrm{i}\pi/q}$, contained in the half-plane $-\frac{1}{2}\pi + \frac{\pi}{q} < \arg\beta^{1/q} < \frac{1}{2}\pi + \frac{\pi}{q}$ of the Riemann surface of the variable $\beta^{1/q}$.

Remark 3.12. Set $\beta' = \beta\mathrm{e}^{-\mathrm{i}\pi}$; then, by theorem 3.11, $E_j(\beta)$ is analytic in the Nevanlinna disc

$$C_{R'} = \left\{\beta : \operatorname{Re}(\beta')^{-1/q} > (R')^{-1}\right\}$$

of the $(\beta')^{1/q}$-plane.

Theorem 3.13. *For any $j \in \mathbb{N}$, the eigenvalue $E_j(\beta)$ of $H(\beta)$ is Borel summable in the ordinary sense for $0 < \arg\beta < \pi$ and in the distributional sense for $\arg\beta = 0$ and $\arg\beta = \pi$.*

Proof. We will examine only the 'singular' cases $\arg\beta = 0, \pi$; the others can be treated in the standard way (see also [7] for $\pi/8 < \arg\beta < 7\pi/8$). Let us consider first the case $\arg\beta = 0$. Then theorem 3.1 allows us to apply the criterion for the distributional Borel–Leroy sum of order q given in [8] and recalled in the appendix for $q = 1$. More precisely, the criterion requires the analyticity of $E_j(\beta)$ in a disc $C_R = \{\beta : \operatorname{Re}\beta^{-1/q} > R^{-1}\}$, as obtained in theorem 3.1, and the well known estimates for the remainders

$$\left|E_j(\beta) - \sum_{s=0}^{N-1} a_s\beta^s\right| \leqslant A\sigma^N\Gamma(qN+1)|\beta|^N \qquad \forall N = 1, 2, \ldots \tag{3.19}$$

uniformly in $C_{R,\epsilon} = \{\beta \in C_R : \arg\beta^{1/q} \geqslant -\pi/2 + \epsilon\}$, $\forall\epsilon > 0$, where the constants A and σ may depend on ϵ, and $\sum_{s=0}^{\infty} a_s\beta^s$ is the Rayleigh–Schrödinger perturbation expansion corresponding to $E_j(\beta)$ (see [20], vol IV, for the standard proof of such estimates). As for the case $\arg\beta = \pi$, we first note that (3.19) is known to hold uniformly in β in any sector

$$S(\delta) = \left\{\beta : |\beta| < B(\delta), -\frac{\pi}{2} + \frac{\delta}{q} < \arg\beta^{1/q} < \frac{\pi}{2} + \frac{\pi}{q} - \frac{\delta}{q}\right\}.$$

Next observe that the direction $\arg\beta = \pi$ in the β-plane corresponds to the direction $\arg\beta' = 0$ in the β'-plane, $\beta' = \beta\mathrm{e}^{-\mathrm{i}\pi}$. Now, in analogy with [8] (theorems 3 and 4), the criterion for the distributional Borel–Leroy summability of order q of $E_j(\beta)$ in the direction $\arg\beta = \pi$ can be stated in terms of the 'adapted' variable β', in the sense that it relies on the following two conditions:

(a) $E_j(\beta)$ is analytic in

$$C_{R'} = \left\{\beta : \operatorname{Re}(\beta')^{-1/q} > (R')^{-1}\right\};$$

(b) $\forall\epsilon > 0$, there exist $A, \sigma > 0$ such that

$$\left| F_j(\beta') - \sum_{s=0}^{N-1}(-1)^s a_s(\beta')^s \right| \leqslant A\sigma^N\Gamma(qN+1)|\beta'|^N \qquad \forall N = 1, 2, \ldots \tag{3.20}$$

uniformly in $C_{R',\epsilon} = \{\beta \in C_{R'} : \arg(\beta')^{1/q} \geqslant -\pi/2 + \epsilon\}$, where

$$F_j(\beta') \equiv \overline{E_j\left(\overline{\beta'}\mathrm{e}^{-\mathrm{i}\pi}\right)} = \overline{E_j(\overline{\beta})}.$$

Now, (a) is given in remark 3.12 and (b) follows from the fact that the sector $S(\delta)$, where (3.19) holds uniformly, can be rewritten in terms of $(\beta')^{1/q}$ as

$$S(\delta) = \left\{\beta : |\beta'| < B(\delta), -\frac{\pi}{2} - \frac{\pi}{q} + \frac{\delta}{q} < \arg(\beta')^{1/q} < \frac{\pi}{2} - \frac{\delta}{q}\right\}.$$

Indeed, since the coefficients a_s of the power series are such that $a_s = 0$ if s is odd, equation (3.19) is equivalent to

$$\left| \overline{F_j(\overline{\beta'})} - \sum_{s=0}^{N-1}(-1)^s a_s(\beta')^s \right| \leqslant A\sigma^N\Gamma(qN+1)|\beta'|^N \qquad \forall N = 1, 2, \ldots \tag{3.21}$$

uniformly in $\overline{C}_{R',\epsilon} = \{\beta \in C_{R'} : \arg(\beta')^{1/q} \leqslant \pi/2 - \epsilon\}$, where $\overline{F_j(\overline{\beta'})} = E_j(\beta)$. Finally, this is equivalent to (b) since the coefficients a_s are real. □

Proof of theorem 1.3. According to the terminology introduced in [8] (see the appendix) about the distributional Borel summability, by (3.19) $E_j(\beta)$ represents the so-called 'upper sum' and $\overline{E_j(\overline{\beta})}$ the 'lower sum' for $\beta \in C_R$; conversely, by (3.20), $E_j(\beta)$ is the lower sum and $\overline{E_j(\overline{\beta})}$ the upper sum for $\beta \in C_{R'}$. More precisely, $E_j(\beta)$ admits for $\beta \in C_R$ the integral representation

$$E_j(\beta) = \frac{1}{q\beta}\int_0^\infty B_j(t + \mathrm{i}0)\,\mathrm{e}^{-(t/\beta)^{1/q}}\left(\frac{t}{\beta}\right)^{-1+1/q}\mathrm{d}t \tag{3.22}$$

and the analogous representation holds for $\overline{E_j(\overline{\beta})}$ with $\overline{B(t + \mathrm{i}0)}$ in place of $B(t + \mathrm{i}0)$. For $\beta \in C_{R'}$ the representation analogous to (3.22) holds in terms of the adapted variable β', i.e.

$$\overline{E_j(\overline{\beta})} = F_j(\beta') = \frac{1}{q\beta'}\int_0^\infty B_j(t + \mathrm{i}0)\,\mathrm{e}^{-(t/\beta')^{1/q}}\left(\frac{t}{\beta'}\right)^{-1+1/q}\mathrm{d}t \tag{3.23}$$

because the odd terms in the power series are identically zero. The distributional Borel sum, which must be real for $\beta \in \mathbb{R}$ since the Rayleigh–Schrödinger perturbation series $\sum_{s=0}^{\infty} a_s\beta^s$ has real coefficients, is given by

$$f_j(\beta) = \tfrac{1}{2}\left(E_j(\beta) + \overline{E_j(\overline{\beta})}\right) \tag{3.24}$$

while, as anticipated in the introduction, the difference

$$d_j(\beta) \equiv 2\mathrm{i}g_j(\beta) = \begin{cases} E_j(\beta) - \overline{E_j(\overline{\beta})} & \beta \in C_R \\ \overline{E_j(\overline{\beta})} - E_j(\beta) & \beta \in C_{R'} \end{cases} \tag{3.25}$$

represents the so-called 'discontinuity', which has zero asymptotic expansion. Now, if $\beta \in \mathbb{R}$, by (3.22) and (3.23) we have

$$E_j(-\beta) = \overline{E_j(\beta)}.$$

It follows that $f_j(\beta) = f_j(-\beta)$ and $g_j(-\beta) = -g_j(\beta)$, i.e. $E_j(\beta)$ and $E_j(-\beta)$ have the same real part and opposite imaginary one. This concludes the proof of the theorem. □

Remark 3.14.

(a) For $\beta \in \mathbb{R}$, it follows from (3.24) and (3.25) that

$$f_j(\beta) = \operatorname{Re} E_j(\beta) \qquad d_j(\pm|\beta|) = \pm 2\mathrm{i} \operatorname{Im} E_j(\pm|\beta|). \tag{3.26}$$

Since $E_j(\beta)$ can be interpreted as a resonance of the problem [11], $f_j(\beta)$ represents the position of the resonance and $|d_j(\beta)|/2$ its width. As in the Stark effect, the distributional Borel summability completely determines the resonance.

(b) In the present case $f_j(\beta)$ and $d_j(\beta)$ admit a further interpretation, since by remark 2.10, $\overline{E_j(\overline{\beta})} = E_j^1(\beta)$, where $E_j^1(\beta)$ represents the jth eigenvalue of $H(\beta)$ for $\operatorname{Im}\beta < 0$. As proved for $E_j(\beta)$, $E_j^1(\beta)$ can be analytically continued to Nevanlinna discs analogous to C_R and $C_{R'}$ across the positive and negative real axis, respectively. Thus,

$$f_j(\beta) = \tfrac{1}{2}(E_j(\beta) + E_j^1(\beta)) \qquad \text{and} \qquad d_j(\beta) = \pm[E_j(\beta) - E_j^1(\beta)]$$

where the + holds for $\beta \in C_R$, and the $-$ for $\beta \in C_{R'}$.

(c) As already recalled, the eigenvalues admit the classical Borel integral representation for $\pi/8 + \eta < \arg\beta < 7\pi/8 - \eta$, $\eta > 0$ [7]. Formulae (3.22) and (3.23) yield their explicit analytic continuation to the regions C_R and $C_{R'}$ across the real axis.

4. Summary

To assess the relevance of the distributional Borel summability proved in this paper, let us first recall the physical intuition behind the odd anharmonic oscillators and summarize the known mathematical results.

The introduction of the odd perturbation changes the shape of the harmonic potential creating a finite barrier and thus destroying the confining nature of the unperturbed potential. Hence the phenomenon of shape resonances should occur: all unperturbed bound states should tunnel into resonances. However, this picture meets serious mathematical difficulties, peculiar to the odd nature of the perturbing potential. First, the differential operator $H(\beta) = p^2 + x^2 + \beta x^{2k+1}$, $\beta \in \mathbb{R}$, $D(H(\beta)) = D(p^2) \cap D(x^{2k+1})$ is not closed and not essentially self-adjoint. Hence the spectrum of its closure is the whole of $\mathbb{C}$. This reflects the incompleteness of the classical motions: any initial condition not inside the well reaches infinity in a finite time. $H(\beta)$ admits infinitely many self-adjoint extensions, each one with a pure-point spectrum (see, e.g., [20], vol II). There is, however, no physical reason to single out a particular extension over any other one. Actually, the physics of the problem has nothing to do with them; neither does perturbation theory, which also in this case yields divergent expansions. The purpose of the papers [7, 11] has been to overcome these difficulties and to put the above physical intuition into the appropriate mathematical framework developed for describing resonances and summing divergent perturbation expansions.

More precisely, first the operator $H(\beta)$ is analysed for $\operatorname{Im}\beta \neq 0$. The results are:

(a) If $\beta \in \mathbb{C}$, $\operatorname{Im}\beta > 0$ (analogous results for $\operatorname{Im}\beta < 0$) the operator family $H(\beta)$ defined on the maximal domain $D(p^2) \cap D(x^{2k+1})$ is closed and has compact resolvents. $\forall j = 0, 1, \ldots$, $H(\beta)$ has exactly one eigenvalue $E_j(\beta)$ near the unperturbed eigenvalue $2j + 1$ of $H(0)$ for $|\beta|$ suitably small.

(b) The function $E_j(\beta)$ is holomorphic for $\operatorname{Im}\beta > 0$, and admits a (many-valued) analytic continuation across the real axis to the (Riemann surface) sector

$$S_1(\delta) = \{\beta : |\beta| < B(\delta), -(2k-1)\pi/8 + \delta < \arg\beta < (2k+7)\pi/8 - \delta\} \qquad \forall \delta > 0.$$

(c) The Rayleigh–Schrödinger perturbation expansion $\sum_{s=0}^{\infty} a_s\beta^s$ near $2j+1$ exists, has the property $a_{2l+1} = 0$, $\forall l \in \mathbb{N}$, and is Borel (more precisely, Borel–Leroy of order $q \equiv (2k-1)/2$) summable to $E_j(\beta)$ for $\pi/8+\eta < \arg\beta < 7\pi/8-\eta$, $\eta > 0$. This implies that if $\operatorname{Re}\beta = 0$ and $|\beta|$ is small $E_j(\beta)$ is real.

The interpretation of $E_j(\beta)$, $\beta \in \mathbb{R}$ as a (limit) resonance ($\operatorname{Re} E_j(\beta)$ location, $\operatorname{Im} E_j(\beta)$ width) is achieved in [11]. A suitable cut-off parametrized by $\Lambda > 0$ is introduced to the effect of approximating the polynomial potential by a dilation analytic potential $V_{\Lambda,\beta}(x)$ preserving the well but tending to $\pm\Lambda\beta$ for $x \to \pm\infty$. The operator $p^2 + V_{\Lambda,\beta}(x)$ has an absolutely continuous spectrum, and admits resonances in the standard sense of dilation analyticity. Each function $E_j(\beta)$, $\beta \in \mathbb{R}$ is a limit of such resonances for $\Lambda \to +\infty$.

The present paper solves the problem of relating directly in a unique way the *real*, divergent perturbation expansion near $2j+1$ to the *complex* resonances $E_j(\beta)$, $\beta \in \mathbb{R}$. It is proved above that $E_j(\beta)$ and the perturbation expansion are on a one-to-one relationship through the distributional Borel summability [8], the technique introduced for dealing specifically with this kind of problems. Hence the *real* perturbation expansion determines *both* the location $\operatorname{Re} E_j(\beta)$ of the resonance and its width $\operatorname{Im} E_j(\beta)$. As in the case of the Stark effect, the location of the resonance is given by the perturbation expansion itself, and the width is still determined (to leading exponential order) by the series through the divergent behaviour of its coefficients.

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Appendix

To make this paper as self-contained as possible, we recall here the criterion for distributional Borel–Leroy summability proved in [8], limiting ourselves to the case $q = 1$, corresponding to the generalization of the classical Nevanlinna–Watson criterion.

Theorem A.1. *Let $f(\beta)$ be bounded and analytic in the Nevanlinna disc $C_R = \{\beta : \operatorname{Re}\beta^{-1} > R^{-1}\}$ and let $f(\beta) = (\phi(\beta) - \overline{\phi(\overline{\beta})})/2$, with $\phi(\beta)$ analytic in C_R and such that*

$$\left|\phi(\beta) - \sum_{s=0}^{N-1} a_s\beta^s\right| \leqslant A\sigma(\epsilon)^N N!|\beta|^N \qquad \forall N = 1, 2, \ldots \tag{A.1}$$

uniformly in $C_{R,\epsilon} = \{\beta \in C_R : \arg\beta \geqslant -\pi/2+\epsilon\}$, $\forall\epsilon > 0$. Then the series $\sum_{s=0}^{\infty}(a_s/s!)u^s$ is convergent for small $|u|$ and it admits an analytic continuation $B(u) = B_1(u) + B_2(u)$, where $B_1(u)$ is analytic in $C_d^1 = \{u : \operatorname{dist}(u, \mathbb{R}_+) < d^{-1}\}$, and $B_2(u)$ is analytic in $C_d^2 = \{u : (\operatorname{Im} u > 0, \operatorname{Re} u > -d^{-1})$ or $|u| < d^{-1}\}$ for some $d > 0$. $B(u)$ satisfies

$$|B(t + \mathrm{i}\eta_0)| \leqslant A'(\eta_0)^{-1}\mathrm{e}^{t/R} \tag{A.2}$$

uniformly for $t > 0$, for any η_0 such that $0 < \eta_0 < d^{-1}$. Moreover, setting $PP(B(t)) = (B(t+\mathrm{i}0) + \overline{B(t+\mathrm{i}0)})/2$, $f(\beta)$ admits the integral representation

$$f(\beta) = \beta^{-1}\int_0^{\infty} PP(B(t))\,\mathrm{e}^{-t/\beta}\,\mathrm{d}t \qquad \beta \in C_R \tag{A.3}$$

i.e. $f(\beta)$ is the distributional Borel sum of $\sum_{s=0}^{\infty} a_s\beta^s$ for $0 < \beta < R$ in the sense of definition 1.1.

Conversely, if $B(u) = \sum_{s=0}^{\infty}(a_s/s!)u^s$ *is convergent for* $|u| < d^{-1}$ *and admits the decomposition* $B(u) = B_1(u) + B_2(u)$ *with the above quoted properties, then the function defined by (A.3) is real-analytic in* C_R *and* $\phi(\beta) = \beta^{-1}\int_0^{\infty} B(t + \mathrm{i}0)\,\mathrm{e}^{-t/\beta}\,\mathrm{d}t$ *is analytic and satisfies (A.1) in* C_R.

Remark A.2. The function $\phi(\beta) = \beta^{-1}\int_0^{\infty} B(t + \mathrm{i}0)\,\mathrm{e}^{-t/\beta}\,\mathrm{d}t$ is called 'the upper sum' and $\overline{\phi(\overline{\beta})} = \beta^{-1}\int_0^{\infty} \overline{B(t + \mathrm{i}0)}\,\mathrm{e}^{-t/\beta}\,\mathrm{d}t$ 'the lower sum' of the series. It follows that, for $\beta > 0$, $f(\beta) = \mathrm{Re}\,\phi(\beta)$. On the other hand, with this method we can single out a unique function with zero asymptotic power-series expansion, that is the 'discontinuity'

$$d(\beta) = \beta^{-1}\int_0^{\infty}(B(t + \mathrm{i}0) - \overline{B(t + \mathrm{i}0)})\,\mathrm{e}^{-t/\beta}\,\mathrm{d}t = \phi(\beta) - \overline{\phi(\overline{\beta})}.$$

Thus, $d(\beta) = 2\mathrm{i}\,\mathrm{Im}\,\phi(\beta)$, for $\beta > 0$.

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